Ray Theory of Gas Dynamic Discontinuities

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A geometric theory of the motion of surfaces of discontinuity is based on the quasilinear algebraic system of generalized Rankine-Hugoniot jump conditions for an ideal gas. Vanishing of a characteristic determinant is necessary for the existence of a nontrivial jump. Geometrical, dynamical, and persistence conditions are applied to the discontinuity of an arbitrary strength, resulting in a set of Hamiltonian equations for the position coordinates and for the space-time normal to the surface. The rays are defined as the integral curves of the Hamiltonian system and generate the singular surface that satisfies the imposed jump conditions.

I. Introduction

GASDYNAMIC discontinuities, e.g., shock waves, vortex sheets, or contact surfaces, serve as internal (free) boundaries. Such boundaries depend on and, in turn, determine unique solutions on both sides of the discontinuity. A convenient representation of the surfaces of discontinuity will not only facilitate their numerical computation, but will facilitate also a study of interactions between such surfaces, of formation, and of propagation of disturbances occurring on, confined to, and carried by such surfaces. The use of computers in gasdynamic research places special demands on the method of description of the motion of singular surfaces. In order to be easily understood by numerical analysts of varied backgrounds, a theory of motion of gasdynamic discontinuities should be coached in a language common to classical mechanics, wave theory, acoustics, and the method of characteristics. Such common language is provided by the ray theory, with rays taken to be those special generators of singular surfaces that satisfy a set of Hamiltonian equations and represent space-time trajectories of points confined to such surfaces.

The theory of singular surfaces of various orders was formalized by Hadamard[1] and may be found in a survey article by Truesdell and Toupin[2] who used the framework of the generalized tensor theory necessary when surface-oriented coordinates are employed. The same framework was used by Anile[3] to derive a geometrical theory according to which weak shock waves propagate along rays and satisfy a transport law. Prasad[4] showed that shock rays of arbitrary strength may be defined to be the characteristic curves of the shock manifold equation, a first-order nonlinear partial differential equaiton. Applications to weak shock waves were given by Ramanathan et al.[5]

In the present work, a ray theory is developed for the motion of all types of gasdynamic singularities, without limitations as to their strength, a theory equally applicable to other branches of continuum mechanics and formulated conveniently for numerical computations of discontinuous flows. The theory is an extension of Whitham's[6] theory of inhomogeneous waves and of the group velocity theory as reviewed by Lighthill.[7] It is an outgrowth of the method of characteristics reformulated and put in the ray form by the present author.[8,9]

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Here, we shall make a general observation that lies at the heart of the geometrical methods in mechanics and, therefore, provides a connection or, better, an analogy between the present work and other well-established theories. This analogy is the reduction of the given system of differential equations, through the use of eigen solutions or wave solutions, to a linear and homogeneous algebraic system. Thus, setting the determinant of the coefficient matrix equal to zero gives the necessary condition for the existence of nontrivial solutions. For example, reduction of the nonlinear partial differential equations of gasdynamics to a linear algebraic system forms the basis of Rusanov's[10] general theory of characteristics. In the present work, the same procedure is applied to the nonlinear Rankine-Hugoniot jump conditions obtained directly from the corresponding differential equations of gasdynamics. This became possible only after the jump conditions were put in the quasilinear form by Kentzer.[11]

An interesting consequence of converting a differential system to an algebraic one is the establishment of a formal one-to-one correspondence between differential operators and algebraic multiplication operators.

II. Necessary Conditions

Jump Conditions

Difficulties in tracking singular surfaces in space-time existed also in the method of characteristics viewed as the theory of singular surfaces of first order. These difficulties were overcome by ray formulation. The steps taken by Kentzer[8,9] consisted of using the algebraic necessary conditions for the existence of a singular surface and conditions for differentiability of the necessary conditions. In the present work, we will follow the same procedure and, first, will consider the algebraic system of jump conditions, impose the necessary conditions for existence of discontinuous solutions analogous to the characteristic determinant, and give linearly independent solution vectors. These results will form the basis for the introduction of rays in Sec. III by differentiation along the singular surface subject to the necessary conditions as constraints. Thus, the special nature of the rays will be revealed.

For simplicity, we shall consider an ideal gas with constant ratio of specific heats γ. It was shown earlier by Kentzer[11] that to the Euler system of quasilinear partial differential equations for mass density ρ, velocity vector $\boldsymbol{u}$, and pressure p, valid in domains where the flow variables are differentiable, e.g.,

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0 \tag{1a}$$

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\frac{1}{\rho}\nabla p=0 \tag{1b}$$

$$\frac{\mathrm{D}p}{\mathrm{D}t}-\frac{\gamma p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}=0 \tag{1c}$$

these correspond to the exact, quasilinear, algebraic jump condition valid on a singular surface $\Sigma(t)$

$$(\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n}-U)[\rho]+\langle\rho\rangle\boldsymbol{n}\cdot[\boldsymbol{u}]=0 \tag{2a}$$

$$(\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n}-U)[\boldsymbol{u}]+\langle 1/\rho\rangle\boldsymbol{n}[p]=0 \tag{2b}$$

$$(\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n}-U)[p]-(\gamma\langle p\rangle/\langle\rho\rangle)(\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n}-U)[\rho]=0 \tag{2c}$$

which may be obtained formally from Eqs. (1) by observing the correspondence between, respectively, the differential operators $\partial/\partial t$ and ∇ operating on differentiable functions $\Psi(t, x)$, and the multiplication operators $-U$ and $\boldsymbol{n}$ operating on the differences between (jumps in) the limiting values of Ψ at the two sides of the discontinuity, the jumps being denoted by $[\Psi]=\Psi_2-\Psi_1$. The symbols $\langle\Psi\rangle$ stand for the arithmetic averages of Ψ across the surface, $\langle\Psi\rangle=\frac{1}{2}(\Psi_2+\Psi_1)$. Proper averaging of the coefficient is discussed by Kentzer.[11] U is the normal propagation (displacement) speed of $\Sigma(t)$ and $\boldsymbol{n}$ the unit spatial normal to $\Sigma(t)$.

The algebraic system (2) is linear and homogeneous in the differences $[\rho]$, $[\boldsymbol{u}]$, and $[p]$, with the coefficients being functions of the various averages and of the components of the space-time normal to the singular surface, $N=(-U,\boldsymbol{n})$. If we introduce the notation

$$w=\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n}-U \tag{3}$$

$$\hat{a}^2=\gamma\langle p\rangle/\langle\rho\rangle=\langle\rho a^2\rangle/\langle\rho\rangle \tag{4}$$

a necessary condition for the existence of a nontrivial jump in flow variables across $\Sigma(t)$ is the vanishing of the determinant χ of the coefficient matrix of Eq. (2), namely,

$$\chi=w^3(w^2-\tilde{a}^2\boldsymbol{n}\cdot\boldsymbol{n})$$

$$\chi=\chi_1\chi_2\chi_3\chi_4\chi_5=w^3(w+\tilde{a}\,|\boldsymbol{n}|)(w-\tilde{a}\,|\boldsymbol{n}|)=0 \tag{5}$$

where

$$\tilde{a}^2=\gamma\langle p\rangle\langle 1/\rho\rangle \tag{6}$$

Thus, if a nontrivial jump occurs, we have for the surface displacement speeds U_α corresponding to the vanishing of the factors χ_α of the determinant χ

$$U_\alpha=\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n},\qquad \alpha=1,2,3$$

$$U_\alpha=\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n}\pm\tilde{a}\,|\boldsymbol{n}|,\qquad \alpha=4,5$$

By standard procedures, we determine five vectors

$$\omega_\alpha=\{[\rho],[\boldsymbol{u}],[p]\}^T$$

as solutions of the homogeneous matrix equation corresponding to the vanishing of each of the five factors of the determinant χ modulo indeterminate factors c_α called the strength. The solution vectors ω_α are not unique and form a set that we will take for convenience to be

$$\omega_1=c_1\{1,0,0\}^T$$

$$\omega_2=c_2\{0,\boldsymbol{q},0\}^T,\qquad \boldsymbol{q}\cdot\boldsymbol{n}=0$$

$$\omega_3=c_3\{0,\boldsymbol{r},0\}^T,\qquad \boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{r}\cdot\boldsymbol{q}=0$$

$$\omega_\alpha=c_\alpha\{\langle\rho\rangle,\ \pm\tilde{a}\boldsymbol{n},\ \gamma\langle p\rangle\}^T,\qquad \alpha=4,5$$

Further information on the properties of the matrix in Eq. (2) may be found in Warming et al.[12]

Material Discontinuities

The triple root of Eq. (5), i.e., $w^3=0$, gives rise to singular surfaces with a continuous normal component of velocity. Since the displacement speed U_α, $\alpha=1,2,3$ also equals the normal component of the velocity of the gas, the singular surface is a material surface.

The first solution vector, ω_1, corresponds to a contact surface across which velocity and pressure are continuous and c_1 is the jump in density.

The solution vectors ω_2 and ω_3 correspond to singular surfaces characterized by a jump in tangential velocity components while the pressure, density, and normal component of velocity remain continuous. Thus, the surface of discontinuity is a vortex sheet.

All three solutions, ω_α, $\alpha=1,2,3$, since they correspond to $w=0$ as the necessary condition, may be superposed. Further, the five-component null vectors ω_α are mutually orthogonal and therefore linearly independent. Jump conditions across an arbitrary material surface are expressible as a particular linear combination of the three independent solutions ω_α, $\alpha=1,2,3$.

Shock Waves

Consider now the case $w\neq 0$ or $\alpha=4,5$. Let $v_n=\boldsymbol{u}\cdot\boldsymbol{n}-U$ be the normal velocity component relative to the shock. In Ref. 13 it was shown that if we omit the subscript $\alpha=4,5$,

$$v_{n_2}v_{n_1}=\hat{a}^2$$

constitutes a generalization of the Prandtl-Meyer relation that, as noted by Prasad,[4] is a necessary condition for the existence of a shock wave. This, and the similar result, namely

$$\langle v_n\rangle^2=\tilde{a}^2$$

provide for a convenient interpretation of the two quantities, $\tilde{a}$ and $\hat{a}$, both of which are surface equivalents of the speed of sound, the first equal in magnitude to the arithmetic and the second to the geometric means of the relative normal velocities with $\tilde{a}\geq\hat{a}$.

There remains the question of the uniqueness of shock solutions, of imposition of the second law of thermodynamics, and of a convenient notation in which the solution corresponding to an expansion shock be automatically rejected and only compressive shocks be used.

Sign Convention

The two signs in front of $\tilde{a}$ in Eq. (5) are at our disposal. We choose to work with $\alpha=4$ and obtain for the displacement speed

$$U_4=\langle \boldsymbol{u}\rangle\cdot\boldsymbol{n}+\tilde{a}$$

We agree to treat U as positive if the displacement velocity is in the direction of the normal $\boldsymbol{n}$. Thus, we interpret the direction of $\boldsymbol{n}$ to be that of propagation of shock waves relative to the fluid, or that $\boldsymbol{n}$ indicates the direction the shock is facing. The direction opposite to $\boldsymbol{n}$ is, therefore, the direction of mass flow relative to the shock. If the subscripts 1 and 2 are taken in chronological order to denote the states of the gas before and after passage of the elements of mass through the shock, $\boldsymbol{n}$ points from side 2 (future) to side 1 (past). For a compressive shock, this implies that

$$p_2 \geq p_1, \qquad \rho_2 \geq \rho_1, \qquad a_2 \geq a_1 \tag{7}$$

The foregoing notation was chosen so that $\alpha = 4$ gives a compressive shock. Initial data decide which side of a singular surface is the high-pressure side. In our notation, we shall agree to denote it by the subscript 2 and point $\boldsymbol{n}$ toward side 1. If $\alpha = 5$, $U_5 \approx -\tilde{a} < 0$ for weak shocks in gas at rest. Clearly, the shock propagates then in the direction opposite to that of $\boldsymbol{n}$, or from side 1 to side 2, while the gas crosses the shock from side 2 (higher pressure) to side 1 (lower pressure). Thus, $\alpha = 5$ corresponds in the adopted notation to an expansion shock ruled out by the second law. Henceforth, we will omit solutions corresponding to $\alpha = 5$, retaining only those for $\alpha = 4$ and dropping the subscript when dealing with shocks.

Entropy Condition

The chosen shock solution corresponding to $\alpha = 4$ was shown in Ref. 13 to satisfy an important geometrical condition, i.e.,

$$U_1 \leq U \leq U_2 \tag{8}$$

where $U_j = \boldsymbol{u}_j \cdot \boldsymbol{n} + a_j$, $j = 1, 2$ are the characteristic displacement speeds defined similarly to U, with the same sign convention, and evaluated in the limit as the shock is approached from side 1 or 2.

Equation (8) states that the shock is slower than a characteristic surface that follows it on the high-pressure side, but faster than the characteristic surface propagating in the same direction as the shock on the lower pressure side. Following Lax,[14] we conclude that every point on the shock, located on either side of it, may be connected with a backward drawn bicharacteristic with the initial data plane. We shall refer to Eq. (8), again following Lax, as the entropy condition.

Condition (8) is necessary for the uniqueness of shock waves. It holds for all shock waves in an ideal gas; it need not be so in a general case, and runaway shock may occur should Eq. (8) be violated.

III. Wave Propagation

Hamiltonian Equations

Let us introduce four-component vectors, hereafter denoted by capital letters, with the corresponding lowercase letter denoting the spatial component:

$$N = (n_t, n_x, n_y, n_z) = (n_t, \boldsymbol{n}) = \text{surface normal}$$

$$X = (t, x, y, z) = (t, \boldsymbol{x}) = \text{position vector}$$

If X is taken to denote the position of a point on a singular surface $\Sigma(t)$ with the normal N, it is convenient to introduce a parametric equation of a curve $C(t)$ lying on $\Sigma(t)$, e.g., $\boldsymbol{x} = \boldsymbol{x}(t)$, so that $X = X(t)$ along an arbitrary curve $C(t)$ on $\Sigma(t)$.

Assuming the surface $\Sigma(t)$ to be differentiable in, at least, a small neighborhood of the point X in question, we may use

$$\frac{\mathrm{d}X}{\mathrm{d}t} \cdot N = 0 \tag{9}$$

as a necessary and sufficient geometrical condition that the curve $X = X(t)$ be everywhere tangent to $\Sigma(t)$. The derivative $\mathrm{d}X/\mathrm{d}t$ may be taken along any differentiable curve $C(t)$ tangent to $\Sigma(t)$, leaving us still with infinitely many choices. There exists, however, a very particular and unique direction of differentiation of interest to us.

We recall that the five factors of the determinant (5) are homogeneous of degree one in components of the normal N with $n_t = -U =$ negative of the surface displacement speed when the space component of N is normalized to unity, $|\boldsymbol{n}| \equiv 1$. By the Euler theorem on homogeneous functions, we write the necessary condition for the existence of a nontrivial singular suface as

$$\chi_\alpha = \frac{\partial \chi_\alpha}{\partial N} \cdot N = 0 \tag{10}$$

Comparing Eqs. (9) and (10), we observe that if the arbitrary direction of differentiation in Eq. (9) is chosen to be

$$\frac{\mathrm{d}X}{\mathrm{d}t} \equiv \frac{\partial \chi_\alpha}{\partial N} = V_\alpha \tag{11}$$

then along the so-defined curve $C_\alpha : X = X(t)$ and $\mathrm{d}X/\mathrm{d}t = \partial \chi / \partial N$, lying on the singular surface $\Sigma(t)$, the necessary conditions for the existence of a singular surface (i.e., dynamical conditions) are satisfied automatically, i.e., when Eq. (11) holds, then $\chi_\alpha = 0$ and $\Sigma(t)$ remains singular.

We may add also the condition of persistence of the singular surface, namely, that besides $\chi_\alpha = 0$, also the following must hold along the curve C_α:

$$\frac{\mathrm{d}\chi_\alpha}{\mathrm{d}t} = \frac{\partial \chi_\alpha}{\partial X} \cdot \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\partial \chi_\alpha}{\partial N} \cdot \frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}X}{\mathrm{d}t} \cdot \left\{ \frac{\partial \chi_\alpha}{\partial X} + \frac{\mathrm{d}N}{\mathrm{d}t} \right\} = 0 \tag{12}$$

Here, we have assumed that the factors of the determinant χ_α, known functions of the dynamical variables and of the surface normal, may be considered to be differentiable functions of X and N on the singular surface. Therefore, it is implied that there exist on the singular surface differentiable functions of position on $\Sigma(t)$

$$\rho = \rho(X), \qquad \boldsymbol{u} = \boldsymbol{u}(X), \qquad p = p(X)$$

and that their limits, and the limits of their derivatives, exist as $\Sigma(t)$ is approached from either side so that their total differentials exist on both sides of $\Sigma(t)$. These are essentially the conditions of the Hadamard lemma (see e.g., Ref. 2, p. 492).

The dynamical persistence conditions, Eq. (12), may be satisfied if either of the two factors vanish identically on $\Sigma(t)$ along C_α, or if the two factors remain orthogonal to each other everywhere along C_α. The latter possibility is not general enough and yields no useful information. The first factor on the right-hand side of Eq. (12), namely $\partial \chi_\alpha / \partial N = V_\alpha$, may be normalized so that its time component equals unity. Thus, $\partial \chi_\alpha / \partial N$ never vanishes so that we may take as a sufficient condition of persistence of a singular surface the vanishing of the second factor in Eq. (12), that is,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{\partial \chi_\alpha}{\partial X} \quad \text{along} \quad \frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\partial \chi_\alpha}{\partial N} = V_\alpha \tag{13}$$

Equations (13) form a Hamiltonian system describing the motion of five kinds of pseudo-particles, representing points confined to an appropriate singular surface, moving with the ray velocity given by Eq. (11), and subject to the generalized forces $-\partial \chi_\alpha / \partial X$ with N playing the role of a pseudo-momentum.

Surface Rays

The four factors of the characteristic determinant [Eq. (5)] of interest to us are of two types, viz. either

$$\chi_\alpha = w = \langle u \rangle \cdot n - U = 0$$

or

$$\chi_4 = w + \tilde{a} |n| = \langle u \rangle \cdot n + \tilde{a} |n| - U = 0$$

the first type pertaining to material surfaces, $\alpha = 1, 2, 3$, and the second to the compression shock. Results for material surfaces may be obtained formally from those for a shock wave by reduction to the case $\tilde{a} = 0$. Without loss of generality, we shall carry out the argument for the compression shock only, with the corresponding results for material surfaces obtained by setting $\tilde{a}$ equal to zero.

In order to obtain the rays, we evaluate the partial derivatives of χ_4 in Eq. (11)

$$\frac{\partial \chi_4}{\partial N} = \left\{ \frac{\partial \chi_4}{\partial (-U)}, \frac{\partial \chi_4}{\partial n} \right\} = \{1, \langle u \rangle + \tilde{a} n\} = V = (1, v) \quad (14)$$

We shall refer to V in Eq. (14) as the ray, a vector tangent to the surface $\Sigma(t)$ and always orthogonal to the surface normal. Thus, a point lying on the surface $\Sigma(t)$ and moving in the direction of the ray V obeys the necessary dynamical condition

$$V \cdot N = -U + \langle u \rangle \cdot n + \tilde{a}(n \cdot n) = 0, \qquad n \cdot n = 1$$

The space component of V, denoted by the lowercase letter v, is

$$v = \langle u \rangle + \tilde{a} n$$

and represents the spatial distance traversed by the shock point in unit time when the point is moving along the ray. Since the units of v are those of velocity, v will be called the ray velocity. The definition of v, Eq. (14), agrees with that of the group velocity, the velocity of a packet of waves [see Ref. 7].

For a vortex sheet, which is a material surface, the normal velocity components on the two sides of it are equal, whereas the tangential components must be averaged, and the ray velocity v may be decomposed into

$$v = (u \cdot n) n - n \times (n \times \langle u \rangle)$$

A contact discontinuity is also a material surface across which velocity (and pressure) are continuous. Thus

$$v = (u \cdot n) n$$

We shall define now the ray to be the Hamiltonian trajectory of a shock point in space-time determined by the Hamiltonian system, Eqs. (13). Since the components of the normal $N = (-U, n)$ of a shock point change along the ray, we need to evaluate the time rate of change of N along the ray, the first of Eqs. (13)

$$\frac{dN}{dt} = -\frac{\partial \chi}{\partial X} = -\left\{ \frac{\partial \chi}{\partial t}, \frac{\partial \chi}{\partial x} \right\}$$

The time component of N is $-U$. Thus

$$\frac{d(-U)}{dt} = -\frac{\partial}{\partial t} (\langle u \rangle \cdot n + \tilde{a} |n|)$$

the differentiation of the right-hand member carried out at constant U and n. The shock acceleration becomes

$$\frac{dU}{dt} = \left\langle \frac{\partial u}{\partial t} \right\rangle \cdot n + \frac{\partial \tilde{a}}{\partial t} \quad (15)$$

where

$$\frac{\partial \tilde{a}}{\partial t} = \frac{1}{2} \tilde{a} \left\{ \frac{1}{\langle p \rangle} \frac{\partial \langle p \rangle}{\partial t} + \frac{1}{\langle 1/\rho \rangle} \frac{\partial \langle 1/\rho \rangle}{\partial t} \right\}$$

The spatial component of N, the unit vector n, may only change in direction. Following Varley and Cumberbatch[15], we set $n = n / |n|$, use index notation, and obtain

$$\frac{dn_i}{dt} = (n_i n_j - \delta_{ij}) \left\{ n_k \left\langle \frac{\partial u_k}{\partial x_j} \right\rangle + \frac{\partial \tilde{a}}{\partial x_j} \right\} \quad (16)$$

where

$$\frac{\partial \tilde{a}}{\partial x_j} = \frac{1}{2} \tilde{a} \left\{ \frac{1}{\langle p \rangle} \frac{\partial \langle p \rangle}{\partial x_j} + \frac{1}{\langle 1/\rho \rangle} \frac{\partial \langle 1/\rho \rangle}{\partial x_j} \right\}$$

and where δ_{ij} is the Kronecker tensor. The derivatives of $\tilde{a}$ are to be set equal to zero for material surfaces.

The ray trajectories reduce to straight lines in space-time only when the flow on both sides of the moving singular surface is steady and uniform. Thus, except for the trivial case of the piecewise uniform flow, singular surface motion is governed by the gradients of the flowfield.

IV. Construction of Solutions

Shock Waves

Represent a discontinuity by a surface generated by a set of curves, $C(t)$, the integral curves of the Hamiltonian equations, Eqs. (13). Given the flow on the low-pressure side (side 1) at $t = t_0$, and the shock surface $\Sigma(t_0)$, its location $x(t_0)$, orientation $n(t_0)$, and velocity $U(t_0)$, the initial conditions on the high-pressure side (side 2) are obtained by satisfying, in a standard way, the shock jump conditions, e.g., by a reduction at each shock point to the stationary normal shock. To accomplish this, subtract from the flow the velocity $Un - n \times (n \times u_1)$ where U, n, and u_1 are presumed given, calculate the state of the gas on side 2 from normal shock relations, and add back the velocity given previously. Then, the rates of change of x, U, and n along the ray may be obtained at $t = t_0$ from Eqs. (14), (15), and (16), respectively.

Now the integration of Eqs. (14-16) from t_0 to $t_1 = t_0 + \Delta t$ gives new position coordinates of shock, $x(t_1)$; its velocity, $U(t_1)$; and orientation, $n(t_1)$. Likewise, the flow on side 1 (low pressure) determines ρ_1, u_1, and p_1 at the new shock location independently of it. The ray form of the method of characteristics[8] could be used here. The solution on side 2 may then be obtained as just presented by reduction to a stationary normal shock. Repeating this procedure advances the shock surface further in time.

Material Surfaces

Initial conditions on a material surface must be specified subject to the constraints that the pressures and normal components of velocity be continuous across an initial discontinuity. Then the displacement speed U is known also. With $\tilde{a} = 0$, the ray equations, Eqs. (14-16), give rates of change of x, U, and n that when integrated to $t_0 + \Delta t$ determine the new location x, orientation n, and displacement speed U. The flow on each side must be likewise advanced in time to $t_0 + \Delta t$ subject to the boundary conditions $p_2 = p_1$, $u_2 \cdot n = u_1 \cdot n = U$. The ray form of the method of characteristics,[8] applied simultaneously to both sides of the discontinuity, will be found convenient for matching the flows subject to the linear boundary conditions on $\Sigma(t)$.

We conclude that as long as Eqs. (14–16) remain valid unique continuation of initial surfaces of discontinuity in time is possible. The continuous dependence of the singular surface motion on the initial data and on the boundary conditions is also evident.

Breakdown of Solutions

Except in the trivial case of piecewise uniform flows, the rays will not remain parallel to each other in space-time. Therefore, it may happen very well that the rays, corresponding to some smooth distribution of the density of points on the initial singular surface, would intersect or form an envelope when carried forward in time. At such an intersection point of the rays, one would obtain a nonunique solution. Thus, a singularity would occur on an initially smooth surface. The assumption of differentiability of the surface in a neighborhood of a point $X(t)$ on the surface $\Sigma(t)$ is thus violated. Such situations may be common in practice and are expected to occur, e.g., in the case of a shock wave converging to form a caustic or in the case of a kink forming on a vortex sheet. The ray formulation of the motion of gasdynamic discontinuities allows one to monitor the behavior of rays and to foresee their impending intersections. How to continue surface motion of solutions displaying singular behavior is the subject of ongoing research. Some numerical techniques have been developed recently. For example, extensive applications of geometrical shock dynamics are reported by Henshaw et al.[16] where Whitman's[6] approximation is used, normal (not Hamiltonian) trajectories of points on weak shocks are determined, and by deleting points, shock-shocks (discontinuities on the shock front) are effectively fitted into the shock front whenever necessary.

The singularities of wave fronts, as well as those of their ray systems, have been studied systematically only recently. A review of the subject and a classification of such systems is given by Arnold.[17]

V. Conclusions

We find the ray formulation of the kinematics of singular surface propagation to be particularly convenient because of the following aspects:

1) Representation of a singular surface at a fixed time by a set of points and in space-time by the trajectories of the set.

2) Development of the singular surface in time governed by the Hamiltonian system of equations familiar from other branches of mechanics.

3) Assured existence and uniqueness of solutions locally, and the possibility of foreseeing the formation of singularities on the singular surfaces themselves.

4) Ease of imposition of exact boundary conditions on moving discontinuities.

5) Imposition of conservation laws and of the second law of thermodynamics.

6) Possibility of generalization of the formulation to other branches of continuum mechanics.

7) Ease of implementation in numerical codes for computation of discontinuous flows.

Generalization to media other than an ideal gas should follow easily because the present technique is directly applicable to algebraic systems of jump conditions that, as shown earlier,[11] may be put in a quasilinear form. Studies of strong interactions between shock waves, vortex sheets, and contact discontinuities, such as the investigation of the remarkable features of structure and stability of supersonic jets,[18] would benefit from the ray formulation.

The present ray formulation establishes also an interesting one-to-one correspondence between the differential and multiplication operators that may, in the future, lead to a statistical theory of gasdynamics based on the random motion of singular surfaces. A statistical gasdynamics in a quantumlike framework, based on an expansion in wave solutions of the locally linearized Navier-Stokes equations, was proposed by Kentzer.[19]

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